**On the possibility of using the TFCV in solving problems of modern logistics**

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**Abstract.** Modern logistics is characterized by the complexity of management objects and the need to process large amounts of data. Therefore, part of scientific research in this direction is aimed at obtaining new methods and building new models that more effectively cope with these problems. The article discusses the possibility and the main problems of using complex-valued economics methods for this. It shows how to generate complex variable models. The main reason is revealed why the section devoted to the processing of a complex random variable is poorly developed in mathematical statistics. These materials will help expand the possibilities of applying methods and models of a complex economy in the practice of logistics.

**Keywords:** Complex random variable, economic models, complex-valued economy, distribution of a complex random variable.

**Introduction**

It is difficult to imagine modern logistics without mathematical methods and models. With their help, numerous tasks are solved - from forecasting demand to optimizing traffic flows. At the same time, practicing specialists are often faced with the fact that the mathematical methods and models at their disposal do not allow obtaining the desired results. Therefore, scientists continue to conduct research in the development of new logistics methods and new models.

One of the interesting and promising areas in this area is the complex-valued economy. By complex-valued economy, we mean a section of economic and mathematical modeling that uses complex variables and models of theory of complex-valued functions.

Any complex number can be considered as a two-dimensional vector. And any complex-valued function can be considered as a function of a two-dimensional vector. Then it is clear that complex variables can be used where two economic indicators reflect different properties of one phenomenon or where the behavior of an object is determined by two factors

That is, where there is a dependence of one two-dimensional vector of economic indicators on another two-dimensional vector of economic indicators (or several two-dimensional indicators), it is very easy to describe such a dependence using models and methods of the complex variable function theory.

Where do such dependencies occur in the economy? Yes, almost at every step, because so many economic indicators are aggregated values that consist of several and, most often, two main values.

For example, gross output *Q* is made up of gross costs *C* and gross profit *G*, which makes it possible to represent the results as a complex variable *(C+iG).*

The capital that is used in production is also an aggregated value – it can be production *K0* and non-production *K1*. Therefore, it can also be represented as a complex variable *(K0+iK1).*

The labor used in production is also a generalized quantity. And it is also possible to take into account different effects of different f labor types on the results of production using a complex variable *(L0+iL1).* Here *L0* is the labor of production personnel, and *L1* is the labor of non-production personnel.

On stock exchanges, analysts monitor the dynamics of different companies` sales shares. In doing so, they use the aggregated value of transactions volume in shares:

. (1)

They will receive additional information if they use a complex variable *(pt+iqt),* having previously reduced the price per unit of *pt* shares and the volume of sales of *qt* shares at this price to a single scale and dimension. This possibility is demonstrated in the monograph "Complex-valued ..." (Svetunkov, 2012).

For the economy as a whole, at the meso or macro level, the volume of production can be divided into goods *P* and services *S* – and again, a complex variable *(P+ iS)* can be used for modeling.

In retail, these can be durable goods *L0* and non-durable goods *Sh*, and again the sales volumes can be described by using a complex variable *(L0 + iSh).*

Similar examples can be continued further, but in any case, it is already clear that the scope of complex variables application in economics is extensive.

In this case, most often it does not matter which variable is attributed to the real part of the complex variable, and which to the imaginary one, because when forming a vector, such a problem is not worth it.

But when modeling the economy, there are such situations when the order of the variables, when they are formed into a complex number, takes on a certain meaning This may be the case, for example, when modeling the dependence of production results on production resources, that is, when constructing production functions.

Large logistic corporations can be described with good accuracy using models of production functions. In the domain of real variables, production functions most often take the form of power models of the type:

. (2)

And their difference from each other lies in the way of setting restrictions on the limits of change in degree indicators αand *β*.

Complex-valued analogues of this function can be very diverse, but the form of a complex-valued production function with real coefficients is universal (Svetunkov S.G., Svetunkov I.S., 2019):

. (3)

Here *C* stands for gross production costs,

*G* - is gross profit of large logistic corporation,

*L* - is labor costs of large logistic corporation,

*K* - capital.

This function can be represented as a system of two real equations:

 or  (4)

It is obvious that in the range of real numbers, production functions of this kind do not occur.

It can be seen from (4) that it is precisely this order of assigning real numbers to the real or to the imaginary parts of complex variables that turns out to be important.

First of all, it should be noted that the real and imaginary parts of the complex production result of the model (3) react differently to the increase in each of the resources. So, with an increase in capital expenditures *K* with a constant expenditure of labor resources *L*, the cosine of the resources angle decreases, and the sine of this angle increases. Then it follows from the first equation of the system (4) that with the growth of capital, gross production costs decrease, and from the second equation of the system (4) it follows that gross profit increases. And that is exactly what happens in the real economy. That is, the complex-valued model (3) describes the details of the production process more accurately than the real model (2).

But here we have several more properties of the production function that are important from an economic point of view. If we present the complex production result in the exponential form of the record, then we get:

. (5)

What characterizes the polar angle of a complex production result? It characterizes the profitability of production (the ratio of gross profit to gross costs). That is, it is this order of attribution of gross profit to the imaginary part, and gross costs to the material part that makes economic sense. And since the polar angle of complex production resources characterizes the capital-labor ratio (the ratio of capital to labor), then the complex-valued production function (3) models the impact of capital-labor ratio on the production profitability. This testifies in favor of the fact that the model (3) has a bright economic sense.

But that is not all! Since the economic meaning of such equality is obvious

, (6)

it turns out that the complex-valued model (3) describes not only the impact of resources on gross costs *C* and gross profit *G*, but also on gross output *Q*.

So, in some cases, the question of which part to attribute a particular economic indicator- to the real or to the imaginary part of a complex number - is important.

In order to use the apparatus of complex variables functions theory in economics when combining two economic indicators into one complex variable, the following conditions must be met, determined by the features of complex numbers:

1. These indicators should be two characteristics of one and the same process or phenomenon, that is, they should reflect different sides of this phenomenon;

2. They should also have the same dimension. In addition, they should have the same scale.

The first condition results from such reasons.

As a result of a complex variable formation from the two real variables, the complex variable is further considered as an independent single variable. Figuratively speaking, it carries information about its two constituent quantities and reflects the influence of each of these components on a certain result. These values must reflect different sides of the same phenomenon, otherwise their combination into one variable loses every meaning. They may be in close functional dependence with each other, or they may have a complex indirect relationship, but the main condition is that they must carry information about some common process for them. This is due to the fact that such characteristics of a complex number as its modulus and argument make sense only when the complex number components reflect the general content.

The second condition, which requires the same dimension of the complex variable components, is determined by the peculiarity of the complex number properties

Indeed, how can the modulus of a complex number be calculated if the real and imaginary parts have different dimensions, for example, rubles and pieces? There is no way to square each of them and add them – rub2 cannot be added to pcs2. Similarly, when calculating the polar angle, it is necessary to find the ratio of the imaginary part to the real part, and then find the arctangent of the resulting number. If the real and imaginary parts are of different dimensions, then nothing can be done, because the tangent of the angle is a dimensionless value, it cannot be measured in rubles / pieces.

In economics, a significant part of the indicators can be reduced to monetary units of measurement, for example, labor costs can be determined not in "man-hours", but in the cost of labor remuneration – by the amount of the wages fund at the enterprise or the enterprise subdivision. Therefore, the condition of the same dimension and scale in most of the real economic problems is quite feasible. But in the case when it is impossible to do this, each of the indicators should be reduced to relative dimensionless values in the way that turns out to be the best for the chosen form of the model.

Hereinafter we will assume that all these conditions are met and we can work with economic random complex variables.

The practical implementation of a complex economy is difficult. The lack of the necessary mathematical apparatus is the reason for this. The mathematical statistics of a complex random variable is a limited branch of mathematical statistics. There are no methods and techniques in it with the help of which it is possible to solve applied problems of logistics. In our opinion, the reason for this situation is a simplified view of the properties of a complex random variable.

**Materials and Methods**

If a researcher has several observations of a complex variable of economic indicators at his disposal, then they can be considered as random variables, since in the economy there are always a lot of random factors.

The complex random variable *Y* is the value

, (7)

in which *yr* and *yi* are real random variables, and *i* is an imaginary unit.

This complex random variable in a stationary process is a point on the complex plane, and the set of observations of it, which is subject to the influence of random factors, on the complex plane will be a certain scattering cloud.

This complex random variable in a stationary process represents a point on the complex plane, and a number of observations of it, which is affected by random factors, on the complex plane will represent some scattering cloud.

All random processes that are studied by the economists have various features, which can mostly be reduced to some typical situations. These typical situations are well described in mathematical statistics and are called "the law of probability distribution." In order to understand which law of probability distribution a particular process belongs to, it is necessary to calculate its main characteristics and draw an appropriate conclusion from them.

The main characteristic of any random variable, including a complex random variable, is its mathematical expectation.

The mathematical expectation of a complex random variable (7) is called a complex number

. (8),

where *mr* is the mathematical expectation of the real part, and *mi* is the mathematical expectation of the imaginary part of a complex random variable.

Since complex random numbers are points on the complex plane, the mathematical expectation of a complex random variable is also a point on the complex plane around which random complex variables are scattered. Moreover, for a normal distribution, such a rule is obvious - the probability that a random complex variable will be closer to its mathematical expectation is greater than the probability that it will be further from it.

The way the points are located on the complex plane is due to the presence or absence of a relationship between the real and imaginary parts of a complex random variable. Let us therefore consider two possible cases:

1) when both parts of a complex random variable are independent of each other. This is the point of view that exists in science today,

2) when the real and imaginary parts of a complex random variable are interrelated. This is the point of view that exists in science today.

The first case. The real and imaginary parts of a complex random variable are independent of each other. In modern mathematical statistics, this position is the main one and is considered as an axiomatic (*Panchev*, 2013). It is quite possible that in those branches of modern science where statistics of a complex random variable are used, this is the case. As the review of published works in this area shows, the complex random variable is mainly used in the signal theory (*Schreier Peter J., Scharf Louis, 20*10; *Steven M.Kay*, 2010; *Tuelay Adili*, *Peter J.Schreier, Louis L. Scharf*, 2011 etc.) and there the independence of two signals of each other is quite a natural phenomenon. And since modern scientists mainly consider this particular case, we cannot ignore it, although it was previously shown that such cases are meaningless for the economy.

Since the real and imaginary parts of such a complex variable do not depend on each other, then all statistical characteristics of a complex random variable do not depend on each other.

Then the variance of the real part of the complex random variable will be equal to:

, (9)

and the variance of its imaginary part is:

. (10)

The total variance of a complex random variable with independent real and imaginary parts will be equal to the sum of its real and imaginary parts variances:

. (11)

It should be noted that variance is an important characteristic that makes it possible to describe the probability distribution density. We will exclusively consider the normal probability distribution, since most often we have to deal with it in practice.

The Gauss formula for the real part of a complex number will look like this:

. (12)

Similarly, we can write down the probability distribution density formula for the imaginary part of a complex number:

. (13)

Then, due to the independence of the real and imaginary parts of a complex random variable from each other, its distribution density will be equal to the product of the distribution densities of the real and imaginary parts:

. (14)

The form of this distribution is shown in Fig. 1, where the axes of the horizontal plane are the real and imaginary parts of a complex random variable (complex plane), and its distribution density is plotted horizontally.

*mY*

*yr*

*0*

*f(Y)*

*yi*

*myr*

*myi*

Figure 1. Mathematical expectation of a complex random variable with independent real and imaginary parts

All points on the complex plane have a different probability of occurrence – the further away from the mathematical expectation of *my*, the less likely they are to appear on the complex plane

All points lying on a straight line with the coordinate *mi* have a different probability of occurrence, and the maximum probability of a random complex variable occurrence falls on the mathematical expectation point *my.* Similarly, all points lying on a straight line with the coordinate *mr* have different probability of occurrence, but the maximum probability of occurrence of a random complex variable on this line also falls on the mathematical expectation point *my*.

It can be seen from the figure that the cross-section of the surface in Fig. 1.1 with planes parallel to the complex plane, that is, planes of equal probability density, gives different ellipses. In probability theory, these ellipses are called "scattering ellipses", the equation of which in our case is determined by the variance of each of the parts and their mathematical expectation:

 (15)

Ellipses can be easily projected onto the complex plane (Fig. 2).

*mY*

*yr*

*0*

*yi*

*myr*

*myi*

.

*yi1*

*yi2*

*1*

*2*

*yr1*

*yr2*

Figure 2. Scattering ellipses for the situation of independence of the real and imaginary parts of a complex random variable.

*1*

In this figure, two points are plotted - 1 and 2, which characterize two different random complex numbers *(yr1; yi1*) and (*yr1; yi1*). If we calculate the distances from them to the mathematical expectation, we get:

 and (16)

It is known that for the discrete case, the variance of a random variable *x* can be written as:

. (17)

For a complex random variable (7), the form of the variance record will be as follows:

. (18).

As it clearly follows from (18), the variance of a complex random variable with real and imaginary parts independent of each other will be the sum of the squared distances from the random variables lying on the complex plane to their mathematical expectation multiplied by the probabilities of these random variables’ occurrence.

If the independent variances of the real and imaginary parts are equal to each other, then the ellipses of Fig. 2 turn into scattering circles.

The second case. The real and imaginary parts of a complex random variable are dependent on each other.

It would seem that in mathematical statistics it was quite logical to study all possible variants of a complex random variable properties. And if there is a variant of the real and imaginary parts of a complex random variable independence from each other, then there should be the second option - the variant of the real and imaginary parts of a complex random variable dependence on each other. Then all the options will be considered and the scientists will get full knowledge about the subject of research.

But it is just the variant of the real and imaginary parts of a complex random variable dependence on each other that the scientists did not consider in full. To all such cases and statistical characteristics, they, thanks to H. Harter and M. Lum`s example, add the prefix "pseudo" (Harter, 1955). And there are historical reasons for this.

Interest in statistical processing of observations of changes in a complex variable arose in the 50s-60s of the twentieth century. For the first time this problem was formulated by R. Wooding, who proposed an approach of a complex random variable representation from the standpoint of a normal distribution (Wooding, 1956). This approach was developed in their works by R. Arens (Arens, 1957) and I. Reed (Reed, 1962). A priori, these publications assumed the independence of the normally distributed real and imaginary parts of a complex random variable from each other, but this was not explicitly mentioned. In 1963, N. Goodman explicitly formulated this assumption (Goodman, 1963). Based on it, scientists further formulated the basic concepts and characteristics of a random normally distributed complex variable: mathematical expectation, moments (including the correlation moment), covariance, variance, etc. (Feller, 1966).

Modern researchers who use complex random variables in their scientific works always use the option when both parts of it are independent of each other (Tavares, 2006). There were, however, the first attempts to comprehend the situation when the real and imaginary parts depend on each other, but scientists considering this option immediately add "pseudo" - pseudo covariance or pseudo variance, etc. (Picinbono, 1997; Kammeyer, 2002; Soroush, 2010; Adali, 2010; Tuelay, 2011 etc.). And they do not go further than calculating pseudo moments, pseudo variances and pseudo covariances.

Therefore, it turned out that we have no ready-made solutions proposed by mathematical statistics for random complex variable whose real and imaginary parts depend on each other. We will have to deal with this issue on our own, checking with the level that modern mathematical statistics offers us in this matter.

Since random variables are considered, the relationship between them will be correlative. Denote by *rri* the coefficient of paired correlation between the real *yr* and the imaginary *yi* part of a complex random variable

The density of the normal distribution of two random interconnected quantities, as is known from the theory of probability and mathematical statistics, taking into account the notation we have adopted, will take the form:

. (19)

One can make sure that when the pair correlation coefficient *ryryi* is equal to zero, the formula (19) turns into the formula (14).

This formula is used in modern mathematical statistics to describe the probabilistic characteristics of normally distributed random complex variables (Trampitsch, 2013, p. 40).

The density of a complex random variable normal distribution with its interconnected parts has in three-dimensional space approximately the same form as shown in Fig. 1, but with a slight difference. As can be seen from Fig. 1, with the independence of the real and imaginary parts of a complex random variable, the three-dimensional model of the distribution density is symmetric with respect to the lines passing through the mathematical expectation point and parallel to the axes of the complex plane.

And in the case of the dependence of these real and imaginary parts of a complex random variable on each other, the model becomes asymmetric to these lines. It becomes symmetrical to the lines that are not parallel to the axes of the complex plane (Fig. 3). In this case, the scattering ellipses also change their position

*mY*

*yr*

*0*

*f(Y)*

*yi*

*myr*

*myi*

Figure 3. Mathematical expectation and probability distribution density with dependent on each other parts of a complex random variable

It is more convenient to consider not a three-dimensional figure in space, but ellipses of scattering.

They, for the case of the existing dependence between the real and imaginary parts, will have the following form

 (20)

Figure 4 shows one of these scattering ellipses.

And it is characteristic for it that the distances from the points lying on the ellipse to the mathematical expectation *mY* are equal

 и  (21)

*mY*

*yr*

*0*

*yi*

*myr*

*myi*

*yi1*

*yi2*

*1*

*2*

*yr1*

*yr2*

Figure 4. The scattering ellipse on the complex plane with the relationship between the real and imaginary parts of a complex random variable

And the variance is respectively equal to:

(22)

Here-  is the probability of a complex random variable occurrence corresponding to (19).

Since the probabilities in the case of a) the independence of the real and imaginary parts and in case of b) their dependences on each other are of different nature and are differently calculated, this means that the variance in the latter case cannot be calculated as in the case of the parts of a complex random variable independence from each other (18), that is:

 (23)

Let us show it.

In Fig. 5, two scattering ellipses corresponding to the same value of the probability distribution density are plotted on the complex plane. The probability of points appearing on the lines of these ellipses is the same. But the blue scattering ellipse corresponds to the situation of the real and imaginary parts of a complex random variable independence from each other, and the red ellipse corresponds to the second variant, when the real and imaginary parts of a complex random variable depend on each other.

*mY*

*yr*

*0*

*yi*

*myr*

*myi*

*yi1*

*yi2*

*1*

*2*

*yr1*

*yr2*

Figure 5. Scattering ellipses on the complex plane with the interrelation between the real and imaginary parts of a complex random variable (red) and in the case of their independence (blue)

Point 1 lies both on the line of the blue scattering ellipse and on the line of the red scattering ellipse. The probability of this point occurrence is the same both in the case of the real and imaginary parts dependence on each other and in the case of their independence from each other. But point 2 lies only on the line of the red ellipse and is above the blue scattering ellipse. This means that the probability *p* of this point occurrence in the case of the real and imaginary parts of a complex random variable independence from each other is less than the probability *pri* of this point occurrence in the case of the components of a complex random variable dependence on each other:

*p< pri*.

Then:

. (24)

Since the variance of a complex random variable with the mutual dependence of the real and imaginary parts is not a simple sum of the variances of the real and imaginary parts, its properties should be studied in more detail.

Complex situation with the use of complex-valued variables models is caused by the fact that the branch of mathematical statistics studying random complex-valued variables seems undeveloped so far (*Svetunkov, 2018*). Scientists paid attention to statistical treatment of complex-valued variables dynamics in the 50-60s of the 20th century. Today this assumption about the autonomy of real and imaginary parts of the complex-valued random variable serves as a key prerequisite for mathematical statistics of complex-valued random variable.

As the scope of functions of modeling using complex-valued variables in various scientific spheres widened, scientists faced the necessity to develop the body of mathematical statistics which allowed to do it. Since this function is of interest not only for economists but for the specialists dealing with complex-valued variables in other sciences as well, an adapted least square method (*Tavares, 2007*) was proposed based on the real part of the complex random value variance. However, at this point the development of the statistical tools of the complex random value stopped – no tools for calculating complex-related correlations were offered, no tools for determining the confidence limits or other instruments of statistical treatment of random complex-valued variables were proposed.

**Results**

Nowadays mathematical statistics deals with variance of independent constituents – real and imaginary parts – rather than variance of the complex-valued variable in general. In this case statistical characteristics of the complex-valued random variable are regarded as real values. To calculate the real characteristics of the complex random value, this value is multiplied by the complex conjugate. This procedure, as it is known, allows to find out the real characteristic of the complex number. Variance of a complex-valued variable is presented as mathematical mean value of squared absolute value of the corresponding centered variable (*Bliss, 2013*; *Panchev*, *2013*; *Steven*, *2010*; *Tuelay, 2011*):

, (25)

where , (26)

. (27)

Viz:

. (28)

However, such interpretation of the complex-valued variable imposes the limits to statistical data treatment. Let us illustrate it by determining the correlations between two complex-valued random variables. Actually, we can calculate the pair correlation coefficient of the variables by using the correlation moment and the variance:

. (29)

Let us take this approach to calculate the correlation between the complex-valued random variables. The correlation moment is represented as a real value using one of the variable in the conjugate form, whereas the variance is calculated as real characteristics in accordance with (25) (*Miyabe*, 2015; *Panchev*, 2013). It should be noted that the correlation moment calculated as

 (30)

is not the real number. It is the complex number as when multiplying and grouping the summands, we obtain:

 (31)

And only in the case when *zX=zY*, the latter summond (31) with the imaginary part equals zero, and the correlation moment becomes a real number. In all other cases the correlation moment will be of complex type, thus the pair correlation coefficient between two complex-valued random variables will be the complex value. Keeping this in mind, scientists claim that they calculate the absolute value of the correlation moment, i.e. instead of (30) they use Re(*µXY*) (*Miyabe*, 2015).

However, let us find the pair correlation coefficient by using (30).

For the sake of simplicity, we assume that the discrete sequence of the complex random value *z* centred in relation to its arithmetic mean is under review, hence:

. (32)

Sample value of the correlation coefficient (29) when using the real-valued variance (25) and the correlation moment (30) is of the form:

. (33)

On the other hand, we should keep in mind that the pair correlation coefficient as related to real numbers was suggested in the 90s of the 19th century by K. Pearson to estimate linear interrelation of complex-valued variables. It was defined as geometric mean of regressions *y* by *x* and *x* by *y* (*Pearson, 2013)*:

, (34)

where the proportionality factors of simple regressions *a1* and *b1* are found by using the least square method.

To find the formula for calculating the sample value of the pair correlation coefficient for the case of the two complex-valued variables using K. Pearson’s approach (34), we shall handle that variant of the least square method which results from the assumption about real character of the complex-valued variable variance (*Tavares, 2007*). The complex regression coefficient of linear relationship between complex-valued variable *Y* and other complex-valued variable *X* by using this approach to the least square method will be calculated in the following way:

. (35)

Inverse relationship of complex-valued variable *X* to other complex random variable *Y* , represented in the linear form, has the following formula for calculating the complex regression coefficient found by using the least square method:

. (36)

By using these formulae of estimating the sample values of proportionality coefficients of regression lines *Y* by *X* and *X* by *Y* in (34), we obtain the formula for calculating the sample value of the complex pair correlation coefficient:

. (37)

Now let us compare formula (33) with formula (37). This should be one and the same coefficient which is calculated based on the same basic prerequisites. However, if the denominators of formulae (32) and (37) coincide, their numerators differ fundamentally from each other. These are different formulae that are used to calculate different coefficients, and these coefficients will give different values for one and the same sequence. This is why it seems unclear: should we use formula (33) or should we use formula (37) or none of these formulae can be used? The obtained result is contradictory, and it does not allow scientists to form the body of complex correlations.

Even if we agree with the suggestions made by the followers of the conception of real-value character of the complex-valued variables variance and use their real parts (*Miyabe, 2013*) instead of complex characteristics, the conflict will not be solved.

In reality, for (33) we will obtain:

, (38)

and for (37) we will obtain:

. (39)

As it can be seen from two results compared, different formulae and a contradictory result are obtained again. This is why P.Schreier and L. Scharf note that so far the research carried out in the area of the correlation of complex-valued random variables has produced deplorable results (*Schreier, 2010)*.

**Discussion**

Exactly the same problems arise in the field of statistical hypotheses and other branches of mathematical statistics of complex-valued variables which to certain extent rely upon an important variability measure – variance. Since economists come across the problem of interrelation (direct or indirect) between the factor and the indicator when considering some object or phenomenon, the assumption that the variance of real and imaginary parts are not interrelated is rarely met. Thus, the hypothesis saying that the variance of the complex random value should always be real cannot be taken as a basis in econometrics. On the contrary, the variance of economic complex-valued random variable should be presented as a complex characteristic of the variability of a random complex sequence (Svetunkov, 2018).

Then the complex variance of the complex random value can be represented in the following way:

, (40)

where 

As will readily be observed, variance (25) is a special case of variance (40), namely – when vectorial angle *θ* between real part and imaginary part of the complex-valued variable is equal to , i.e. real part and imaginary part are not interdependent.

How can the assumed hypothesis about the relationship between the real and imaginary parts, and that the variance of the complex-valued variable should be considered as the complex value, help in solving applied econometric problems? To answer this question let us turn back to the calculation of correlations between complex random values using two methods, which resulted in an impasse if we assume that the variance of the complex-valued variable is a real number.

We shall consider all the characteristics of the complex random value as complex numbers. This is why we shall not resort to their artificial transformation into real numbers of these characteristics by multiplying the complex number by its conjugate. Let us represent the correlation moment of two random complex-valued variables as a complex number:

 (41)

If we apply the values of complex variance (38) and complex correlation moment (41) to the formula for calculation of pair correlation coefficient (29), we obtain:

. (42)

The obtained formula for the calculation of the sample value of the pair correlation coefficient of two random variables (31) does not coincide with any of the previously derived formulae (33) and (37), when all the characteristics were considered to be real ones.

We shall calculate the complex pair correlation coefficient using the second method – as the geometric mean of sample values of the regression coefficients. In order to find this coefficient, let us formulate the criterion of least square method. It is to be recalled that assuming that the variance of the random complex-valued variable is a real value, the least square method means in fact searching for such regression coefficients whereby:

. (43)

where - are approximation errors.

In case of the complex variance of the complex random value the least square method reduces to searching for other coefficients whereby (*Svetunkov, 2012, p. 83*):

. (44)

One can pay attention here to the interrelation between criteria (43) and (44). With this aim in view let us present the complex approximation error in the exponential form:

. (45)

Taking it into account, criterion (43) takes the form:

, (46)

and criterion (44) is as follows:

, (47)

Therefore, criterion of the least square method (43) suggested by G.N.Tavares and L.M.Tavares, is a special case of criterion (44) – when the vectorial angle of the complex approximation error is equal to zero.

Now, using criterion (44) in relation to the complex regression coefficient of complex number *X* by complex number *Y* denoted as *a,* we will obtain such formula using the least square method with criterion (44) (*Svetunkov, 2012, p. 103 – 112*):

. (48)

The complex coefficient of proportionality *b* of the inverse regression can also be calculated by using criterion (44) as:

. (49)

Now when plugging these coefficients in the formula for calculation of the pair correlation coefficient (34), we obtain:

. (50)

As it can be seen, the same formula of the complex pair correlation coefficient (42) as in the case of its calculation through the complex correlation moment (41) is obtained. This means that the obtained result is not contradictory. Both pair correlation coefficient (42) between two random complex-valued variables calculated through variance and correlation moment and the pair correlation coefficient calculated through the geometric mean of linear regression have one and the same form. This in turn means that our hypothesis about the need to use complex variance and other complex characteristics of the complex-valued variables in statistics of the complex-valued random variables, is confirmed.

**Conclusions**

Let us turn now to the analysis of the properties of complex variance of complex random value (38), the use of which in econometrics of the complex-valued random variables has just been justified. To illustrate it, let us write complex variance in the arithmetic form:

. (51)

Depending on what form the real and imaginary parts have, complex variance can be a complex, real or imaginary value – the variety of its values correspond to the variety of the properties of the complex-valued random variable. In addition, complex variance can be both positive and negative. Let us consider these options and properties of the complex random value sequence, for which these options of complex variance are valid.

Firstly, let us pay attention to the imaginary part of complex variance (51):

. (52)

It has a simple meaning – it is double covariance between the real and imaginary parts of the random complex-valued variable. If there is no correlation between variables, the variable covariance is equal to zero. This means that the imaginary part of the complex variance serves the basis to assume on the presence or absence of correlation between the real and imaginary parts of the random complex-valued variable.

The real part of complex variance of the random complex value is also meaningful for the researcher:

. (53)

As it can be seen, it characterizes the degree of distinction between the variance of the real part of the random complex-valued variable and the variance of the imaginary part of the given variable. This is why in case when both types of variance are equal to each other, the real part of the complex variance is equal to zero. If the variance of the real part of the complex-valued variable is larger than that of the imaginary part of the complex-valued variable, real part (51) of the complex variance will be positive. Otherwise, it will be negative.

It is noteworthy that justifying of complex character of the complex random value does not refuse the possibility to use variance in real form – it can be applied as an additional characteristic of the process under research, since real variance characterizes the variability measure of the absolute value of complex variance.

Now we have removed all those artificial restrictions that hindered the development of mathematical statistics of a complex random variable. And now, with the help of this new approach, it is possible to actively use the methods and models of complex-valued economics to solve problems in modern linguistics.

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